Summary 5: Vectors and Coordinate Systems

1. Coordinate Systems

Many aspects of physics involve a description of a location in space. In Summary 1 and 3, for example, we saw that the mathematical description of an object's motion requires a method for describing the object's position at various times. In two dimensions, this description is accomplished with the use of the Cartesian coordinate system, in which perpendicular axes intersect at a point defined as the origin as shown in Figure 1.

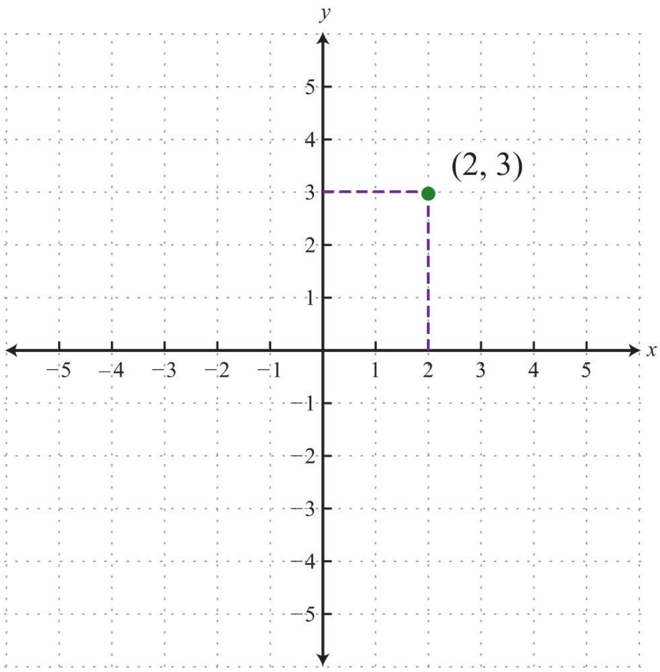

Figure 1. Coordinate system in two dimensions

Sometimes it is more convenient to represent a point in a plane by its plane polar coordinates (r, θ) as shown in Figure 2.

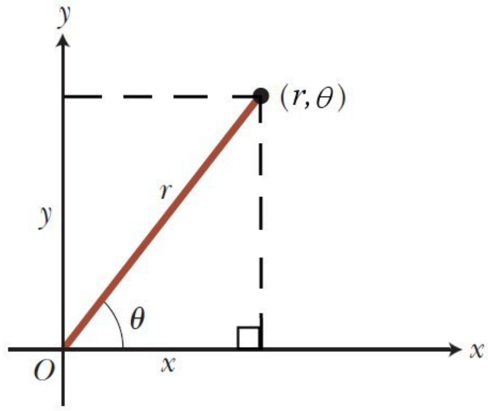

Figure 2. Polar coodinates

In this polar coordinate system, r is the distance from the origin to the point having Cartesian coordinates (x, y) and θ is the angle between a fixed axis and a line drawn from the origin to the point. The fixed axis is often the positive x axis, and θ is usually measured counterclockwise from it. From the right triangle in Figure 2, we find that $sin\theta = {}^{y}/_{r}$ and that $cos\theta = {}^{x}/_{r}$. Therefore, starting with the plane polar coordinates of any point, we can obtain the Cartesian coordinates by using the equations

$$x = rcos\theta \tag{1}$$
$$y = rsin\theta \tag{2}$$

Furthermore, if we know the Cartesian coordinates, the definitions of trigonometry tell us that

$$tan\theta = \frac{y}{x} \tag{3}$$

$$r = \sqrt{x^2 + y^2} \tag{4}$$

Equation 4 is the familiar Pythagorean theorem.

Example 1: Find the polar coordinates of the point A(2,5).
Answer:

$$r = \sqrt{x^2 + y^2} = \sqrt{2^2 + 5^2} = \sqrt{29} = 5.385$$

$$tan\theta = \frac{y}{x} = \frac{5}{3} = 1.67 \Rightarrow \theta = 59\,\text{deg}$$

2. Vectors and scalar quantities

Quantities such as force or velocity, which have a direction as well as a magnitude are called vectors.

Quantities such as mass and volume, which have a magnitude but no direction are called scalars. So:

A scalar quantity is completely specified by a single value with an appropriate unit and has no direction. While a vector quantity is completely specified by a number with an appropriate unit plus a direction.

Another example of a vector quantity is displacement, as you know from Summary 1. Suppose a particle moves from some point A to some point B along a straight path as shown in Figure 3.

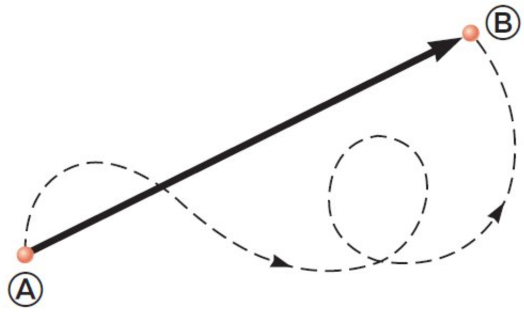

Figure 3. Displacement vector

We represent this displacement by drawing an arrow from A to B, with the tip of the arrow pointing from A to B. The direction of the arrowhead represents the direction of the displacement, and the length of the arrow represents the magnitude of the displacement. If the particle travels along some other path from A to B such as shown by the dashed line in Figure 3, its displacement is still the arrow drawn from A to B. Displacement depends only on the initial and final positions, so the displacement vector is independent of the path taken by the particle between these two points.

3. Adding vectors

The parallelogram rule is a method of finding the resultant of two vectors in situations where the vectors are collinear. For example, as shown in figure 4, if two forces are acting at a point O, we complete the parallelogram made by the two forces, and the resultant is the vector along the diagonal drawn from O having direction away from O and length as that of the diagonal.

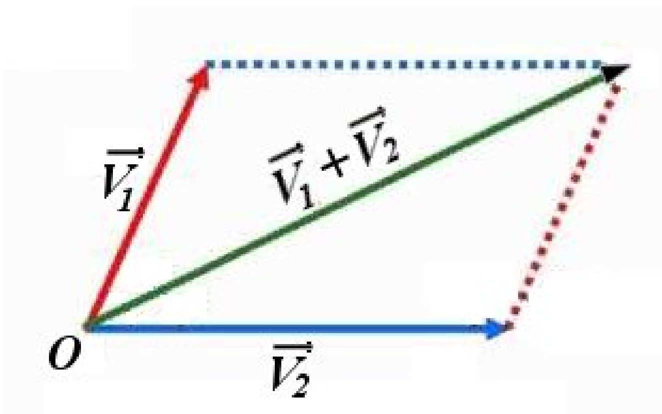

Figure 4. Parallelogram rule

4. Components of a vector

The parallelogram rule also works in reverse: a single vector can be replaced by two vectors having the same effect. Scientifically speaking, a single vector can be resolved into two components. In figure 5, you can see some of the ways in which a 60N force can be resolved into two components. There are endless other possibilities.

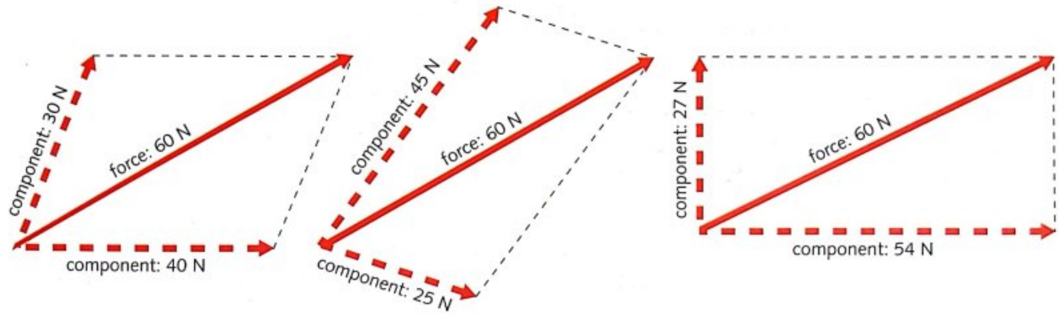

Figure 5. Several ways of resolving a vector

4.1. Scalar components of a vector

In this section, we describe a method of adding vectors that makes use of the projections of vectors along coordinate axes. These projections are called the components of the vector or its rectangular components. Any vector can be completely described by its components.

Consider a vector $\vec{A}$ lying in the xy plane and making an angle θ with the positive x axis as shown in Figure 6-a. This vector can be expressed as the sum of two component vectors $\vec{A}_x$, which is parallel to the x axis, and $\vec{A}_y$, which is parallel to the y axis. From Figure 6-b, we see that the three vectors form a right triangle and that $\vec{A} = \vec{A}_x + \vec{A}_y$

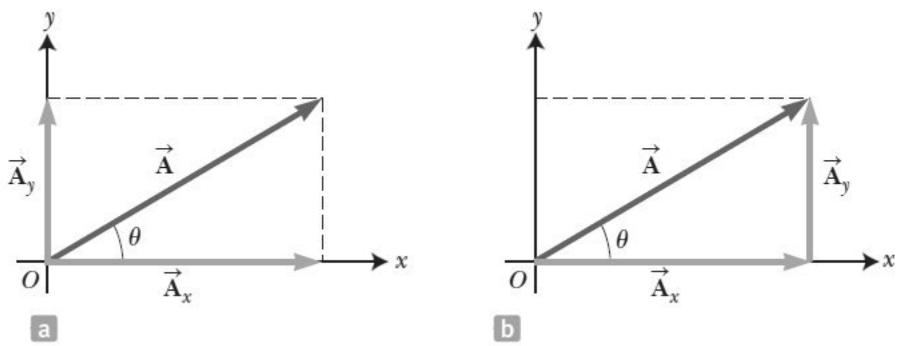

Figure 6. Components of a vector

From Figure 6 and the definition of sine and cosine, we see that $cos\theta = {}^{A_x}/_{A}$ and that $sin\theta = {}^{A_y}/_{A}$. Hence, the components of $\vec{A}$ are:

$$A_x = Acos\theta \tag{5}$$

$$A_y = Asin\theta \tag{6}$$

The magnitude and direction of $\vec{A}$ are related to its components through the expressions:

$$A = \sqrt{A_x^2 + A_y^2} \tag{7}$$

$$\theta = tan^{-1}\left(\frac{A_y}{A_x}\right) \tag{8}$$

Example 2: A vector $\vec{A}$ of magnitude (size) 10 cm makes an angle 40^o with the x-axis. Find its scalar components.
Answer:

$$A_x = Acos\theta = 10 \times cos40^o = 7.66cm$$

$$A_y = Asin\theta = 10 \times sin40^o = 6.43cm$$

Example 3: Find the magnitude and the direction of the vector $\vec{A} = 3\vec{i} + 5\vec{j}$.
Answer: The magnitude of $\vec{A}$ is $A = \sqrt{A_x^2 + A_y^2} = \sqrt{3^2 + 5^2} = \sqrt{34} = 5.83$
The direction of $\vec{A}$ is given by the angle it makes with the x-axis: $\theta = tan^{-1}\left(\frac{A_y}{A_x}\right) = tan^{-1}\left(\frac{5}{3}\right) = 59\,\text{deg}$

4.2. Unit vectors

Vector quantities often are expressed in terms of unit vectors. A unit vector is a dimensionless vector having a magnitude of exactly 1. Unit vectors are used to specify a given direction and have no other physical significance.

We define the unit vectors $\vec{i}$ and $\vec{j}$ in the positive x and positive y directions respectively. Consider a vector $\vec{A}$ lying in the xy plane as shown in Figure 7.

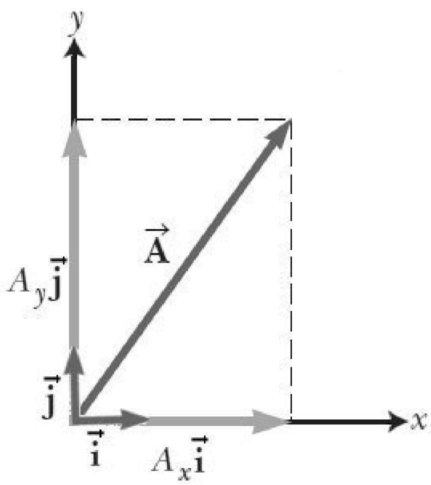

Figure 7. Unit vectors on the x and y axes

The component vector $\vec{A}_x$ can be expressed as $\vec{A}_x = A_x\vec{i}$, and the component vector $\vec{A}_y$ can be expressed as $\vec{A}_y = A_y\vec{j}$. So the vector $\vec{A}$ can be expressed as:

$$\vec{A} = A_x\vec{i} + A_y\vec{j} \tag{9}$$

4.3. Addition of vectors using scalar components

Consider two vectors $\vec{A} = A_x\vec{i} + A_y\vec{j}$ and $\vec{B} = B_x\vec{i} + B_y\vec{j}$. The resultant of the two vectors is $\vec{R} = \vec{A} + \vec{B} = \left(A_x\vec{i} + A_y\vec{j}\right) + \left(B_x\vec{i} + B_y\vec{j}\right) = (A_x + B_x)\,\vec{i} + \left(A_y + B_y\right)\vec{j}$.

Example 4: Consider two vectors $\vec{A} = 2\vec{i} - 5\vec{j}$ and $\vec{B} = 3\vec{i} + 2\vec{j}$.
The sum of these two vectors is $\vec{R} = (2+3)\,\vec{i} + (-5+2)\,\vec{j} = 5\vec{i} - 3\vec{j}$.
The magnitude of the resultant vector is $R = \sqrt{R_x^2 + R_y^2} = \sqrt{5^2 + (-3)^2} = \sqrt{34} = 5.83$.
The direction of $\vec{R}$ is given by the angle θ that $\vec{R}$ makes with the x-axis: $\theta = tan^{-1}\left(\frac{R_y}{R_x}\right) = tan^{1}\left(\frac{-3}{5}\right) = -30.96\,\text{deg}$

5. Some operations on vectors

5.1. Equality of two vectors

Two vectors $\vec{A}$ and $\vec{B}$ may be defined to be equal if they have the same magnitude and if they point in the same direction. That is, $\vec{A} = \vec{B}$ only if $A = B$ and if $\vec{A}$ and $\vec{B}$ point in the same direction along parallel lines, as in figure 8.

Using scalar components of the vectors we can write:

$$\vec{A} = \vec{B} \Rightarrow A_x = B_x \; and \; A_y = B_y \tag{10}$$

For example, the two vectors $\vec{A} = 2\vec{i} + 3\vec{j}$ and $\vec{B} = 2\vec{i} + 3\vec{j}$ are equal.

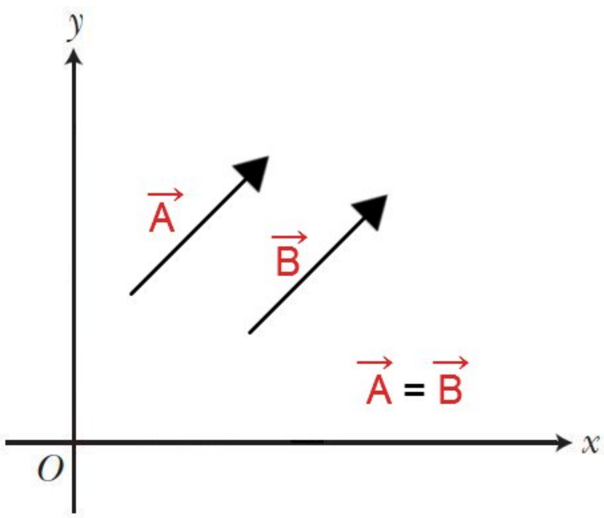

Figure 8. Equal vectors

5.2. Adding vectors

We have already discussed the addition of two vectors using the parallelogram method, but there is another method which is the consecutive vectors method. In this method we translate one of the vectors so that its head is on the tail of the other. The resultant vector is the vector drawn from the tail of the first vector to the head of the second vector as shown on the Figure 9.

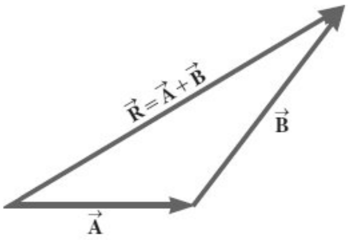

Figure 9. Adding two vectors

We can also use this method to add more than two vectors as shown in Figure 10.

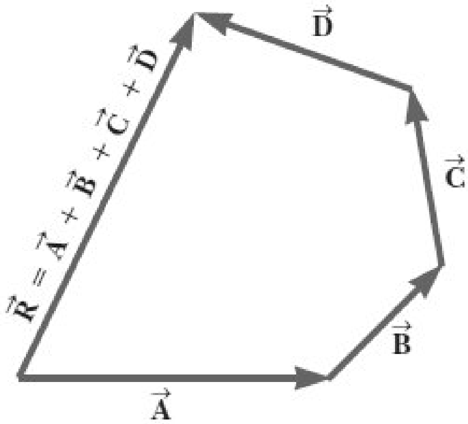

Figure 10. Adding consecutive vectors

In all cases, we can add vectors using their scalar components as discussed earlier in section 4.3.

Example 5: Consider the four vectors $\vec{A} = 2\vec{i} - 5\vec{j}$, $\vec{B} = 3\vec{i} + 2\vec{j}$, $\vec{C} = 3\vec{i} + 4\vec{j}$, and $\vec{D} = -5\vec{i} + 3\vec{j}$.
The sum of these vectors is $\vec{R} = (2+3+3-5)\,\vec{i} + (-5+2+4+3)\,\vec{j} = 3\vec{i} + 4\vec{j}$.

5.3. Negative of a vector - opposite vectors

The negative of the vector $\vec{A}$ is defined as the vector that when added to $\vec{A}$ gives zero for the vector sum. That is, $\vec{A} + \left(-\vec{A}\right) = \vec{0}$. The vectors $\vec{A}$ and $-\vec{A}$ have the same magnitude but point in opposite directions. We also say that $\vec{A}$ and $-\vec{A}$ are opposite vectors. In terms of scalar components, the opposite or negative of $\vec{A} = A_x\vec{i} + A_y\vec{j}$ is $\vec{B} = -\vec{A} = -A_x\vec{i} - A_y\vec{j}$.

For example, the opposite of $\vec{A} = 3\vec{i} - 4\vec{j}$ is $\vec{B} = -\vec{A} = -3\vec{i} + 4\vec{j}$.

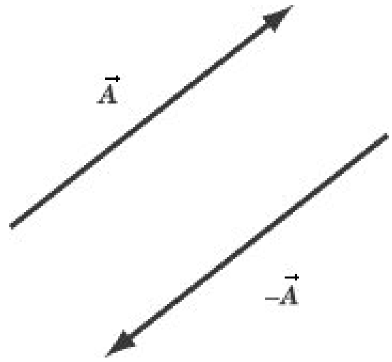

Figure 11. Opposite vectors

5.4. Subtraction of vectors

To subtract a vector $\vec{B}$ from a second vector $\vec{A}$ graphically, first we translate one of the vectors so that the two vectors are placed tail to tail as in figure 12. Then the vector difference $\vec{C} = \vec{A} - \vec{B}$ is the vector drawn from the head of $\vec{B}$ to the head of $\vec{A}$, as shown in figure 12.

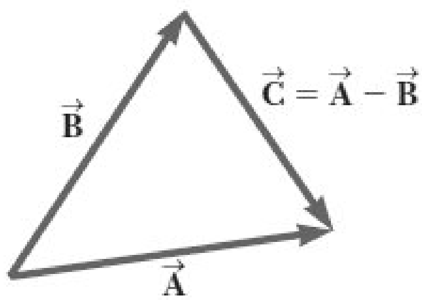

Figure 12. Vector difference

In terms of scalar components, if $\vec{A} = A_x\vec{i} + A_y\vec{j}$ and $\vec{B} = B_x\vec{i} + B_y\vec{j}$, then $\vec{C} = \vec{A} - \vec{B} = (A_x - B_x)\vec{i} + (A_y - B_y)\vec{j}$.

5.5. Multiplying a vector by a scalar

If vector $\vec{A}$ is multiplied by a positive scalar quantity m, the product $m\vec{A}$ is a vector that has the same direction as $\vec{A}$ and magnitude mA. If vector $\vec{A}$ is multiplied by a negative scalar quantity $-m$, the product $-m\vec{A}$ is directed opposite to $\vec{A}$. For example, the vector $4\vec{A}$ is four times as long as $\vec{A}$ and points in the same direction as $\vec{A}$; the vector $-\frac{1}{3}\vec{A}$ is one-third the length of $\vec{A}$ and points in the direction opposite to $\vec{A}$.